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A Localization Principle for Orbifold Theories

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Dedicated to Prof. Samuel Gitler on the occasion of his 70th birthday.

Abstract. In this article, written primarily for physicists and geometers, we survey several manifestations of a general localization principle for orbifold theories such as K-theory, index theory, motivic integration and elliptic genera.

1. Orbifolds

In this paper we will attempt to explain a general localization principle that appears frequently under several guises in the study of orbifolds. We will begin by reminding the reader what we mean by an orbifold.

The most familiar situation in physics is that of an orbifold of the type $\mathsf{X} = [M/G]$, where M is a smooth manifold and G is a finite group acting[1] smoothly on M; namely, we give ourselves a homomorphism $G \to \mathrm{Diff}(M)$. We make a point of distinguishing the orbifold $\mathsf{X} = [M/G]$ from its quotient space (also called orbit space) $X = M/G$. As a set, as we know, a point in X is an orbit of the action: that is, a typical element of M/G is $\mathrm{Orb}(x) = \{xg \mid g \in G\}$.

For us an orbifold $\mathsf{X} = [M/G]$ is a smooth category[2] (actually a topological groupoid) whose objects are the points of M, $\mathsf{X}_0 = \mathrm{Obj}(\mathsf{X}) = M$, and we insist on remembering that $\mathsf{X}_0 = \mathrm{Obj}(\mathsf{X})$ is a *smooth manifold*. The arrows of this category are $\mathsf{X}_1 = \mathrm{Mor}(\mathsf{X}) = M \times G$ again thinking of it as a smooth manifold. A typical arrow in this category is

$$x \xrightarrow{(x,g)} xg,$$

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[1]We will consider mostly right actions. Thus, instead of writing gx for the action of g in x we will write xg, the action being $(x, g) \mapsto xg$.

[2]While this may sound slightly far-fetched at first, tolerating this definition pays off handsomely in simplifying several arguments.

and the composition of two arrows looks like

$$x \xrightarrow{(x,g)} xg \xrightarrow{(xg,h)} xgh.$$
$$\underbrace{\qquad\qquad\qquad}_{(x,gh)}$$

As we have already pointed out, an important property of this category is that it is actually a groupoid: indeed, every arrow (x, g) has an inverse (depending smoothly on (x, g)), to wit $(x, g)^{-1} = (xg, g^{-1})$.

To be fair, the definition of an orbifold is somewhat more complicated. First, we must impose some technical conditions on the groupoids that we will be working with. Second, we must consider an equivalence relation (usually called *Morita equivalence*, related to equivalence of categories) on the family of all smooth groupoids. Then one can roughly say that an orbifold is an equivalence class of groupoids [**34, 29**]. Choosing a particular groupoid to represent an orbifold is akin to choosing coordinates for a physical system, and clearly the theories we are interested in should be invariant under such freedom of choice.

For example, consider the manifold $N = M \times \mathbb{Z}_2$ consisting of two disjoint copies of M, and the group $H = G \times \mathbb{Z}_2$, and let H act on N by the formula

$$(m, \epsilon_0) \cdot (g, \epsilon_1) = (mg, \epsilon_0\epsilon_1).$$

Then not only are $N/H \cong M/G$ homeomorphic, but moreover $\mathsf{X} \cong [N/H] \cong [M/G]$ are equivalent groupoids, while clearly $N \neq M$ and $H \neq G$.

From now on we will be rather cavalier about this issue and always choose a groupoid to represent an orbifold. All our final results will be independent of this choice.

To a groupoid X we will often need to associate a useful (infinite-dimensional) space, denoted by $B\mathsf{X}$, called its classifying space. We start by drawing a graph, putting a vertex for every object $m \in \mathsf{X}_0$ (remembering the topology of the space of objects). Then we draw an edge for every arrow $\alpha \in \mathsf{X}_1$. We fill in with a 2-simplex $\Delta^2 = \{(x, y, z)|x + y + z = 1,\ x, y, z \geq 0\}$ every commutative triangle in the graph (with edges $(\alpha, \beta, \alpha \circ \beta)$). Then we fill with a 3-simplex any commutative tetrahedron, and so on (see [**37**]). There is always a canonical projection map $\pi_{\mathsf{X}} \colon B\mathsf{X} \to X$ so that $\pi_{\mathsf{X}}^{-1}(\mathrm{Orb}(x)) \simeq BG_x$, where G_x is the stabilizer of x, that is, the subgroup of G fixing x.

For example, if the orbifold in question is $\mathsf{X} = [*/G]$, a group acting on a single point, then $B\mathsf{X} = BG$ is the classifying space of G. This space has the property that the set of isomorphism classes of principal G-bundles $\mathrm{Bun}_G(Y)$ over Y is isomorphic to $[Y, BG]$, the set of homotopy classes of continuous maps from Y to BG. When $\mathsf{X} = [M/G]$, it is customary to write $B\mathsf{X} = M \times_G EG$, also known as the *Borel construction of the group action $M \times G \to M$*. For a general groupoid X, we refer the reader to [**32**] for a description of what exactly $B\mathsf{X}$ classifies. It is important to remark that $B\mathsf{X}$ depends on which representation one considers for X, but its *homotopy type* is independent of this choice.

A final remark: there are orbifolds X that *cannot* be represented by a groupoid of the form $[M/G]$. In other words, in spite of the fact that there is indeed a groupoid representing X, nevertheless there is no manifold M with a *finite* group action G so that $\mathsf{X} \cong [M/G]$. We say in this situation that the orbifold in question is *not* a *global quotient*. Examples are given by the toric orbifolds $\mathsf{W}(a_0, \ldots, a_n)$ whose quotient spaces are the weighted projective spaces $\mathbf{P}(a_0, \ldots, a_n)$ (here a_i are

coprime positive integers). For simplicity, let us discuss the case of the orbifold $\mathsf{W}(1,2)$ whose quotient space is the weighted projective line $\mathbf{P}(1,2) \cong \mathbf{P}^1$. One way to describe $\mathsf{W}(1,2)$ is through the system of local charts:

$$\begin{tikzcd} & {[\mathbb{C}^\times/\{1\}]} \arrow[dl, "z\mapsto 1/z^2"'] \arrow[dr, "z\mapsto z"] & \\ {[\mathbb{C}/\mathbb{Z}_2]} & & {[\mathbb{C}/\{1\}].} \end{tikzcd}$$

If $\mathsf{W}(1,2)$ were Morita equivalent to a groupoid $[M/G]$, then this would induce a homomorphism $\rho\colon G \to \mathbb{Z}_2$ (this follows by looking at the unique point in $\mathsf{W}(1,2)$ with isotropy $\mathbb{Z}_2$). Therefore the orbifold $[M'/\mathbb{Z}_2]$ with $M' := M/ker(\rho)$ would be equivalent to $\mathsf{W}(1,2)$. But this is a contradiction because any action of $\mathbb{Z}_2$ in a compact surface cannot have only one fixed point.

This example might be a source of misunderstanding because weighted projective spaces are indeed quotient varieties of manifolds by actions of finite groups. For instance, in our example, $\mathbf{P}(1,2)$ is isomorphic to the quotient of $\mathbf{P}^1$ by $\mathbb{Z}/2\mathbb{Z}$ under the action $[x,y] \mapsto [x,-y]$ in homogeneous coordinates. On the other hand, although the orbifold $\mathsf{W}(1,2)$ can be presented as a quotient of a manifold by an action of a Lie group, namely $[\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$ with $\lambda\cdot(x,y) \mapsto (\lambda^2 x, \lambda y)$, it is not equivalent to global quotient by a finite group. It is worth pointing out that it is still an open question whether every compact orbifold can be presented (up to Morita equivalence) as the quotient of a manifold by a Lie group [**20**].

2. Orbifold K-Theory

In their seminal paper [**17**], Dixon, Harvey, Vafa, and Witten defined the orbifold Euler characteristic of an orbifold $\mathsf{X} = [M/G]$ by the formula

$$\chi_{\mathbf{Orb}}(\mathsf{X}) = \frac{1}{|G|}\sum_{gh=hg}\chi(M^{g,h}), \tag{2.0.1}$$

where (g,h) runs through all the pairs of commuting elements of G and $M^{g,h}$ is the set of points in M that are fixed both by g and by h. They obtained this formula by considering a supersymmetric string sigma model on the target space M/G and noting that in the known case in which $G=\{1\}$ the Euler characteristic of $\mathsf{X} = M$ is a limiting case (over the worldsheet metric) of the partition function on the 2-dimensional torus.

In essentially every interesting example, the *stringy orbifold Euler characteristic* $\chi_{\mathbf{Orb}}(\mathsf{X})$ is not equal to the ordinary Euler characteristic of the quotient space $\chi(X)$. More interestingly, $\chi_{\mathbf{Orb}}(\mathsf{X})$ is truly independent of the particular groupoid representation, namely if $\mathsf{X} = [M/G] \cong [N/H]$ then it doesn't matter which representation one uses to compute $\chi_{\mathbf{Orb}}(\mathsf{X})$. In other words, this is a truly physical quantity independent of the choice of coordinates. This last remark, which can be readily verified by the reader, is quite telling, since *a priori* the sigma model depends on the particular groupoid representation. But as the theory is indeed physical, the final partition function is independent of the choice of coordinates.

Moreover, since the partition function of the theory is physical, one may expect a stronger sort of invariance. Should there be a well-behaved (smooth) resolution of X defining the same quantum theory, then one should have that the Euler characteristic of the resolution is the same as that of the original orbifold. Here we

are shifting our point of view, thinking of an orbifold as the quotient space with a mild type of singularities. It is a remarkable fact in algebraic geometry [**11**] that in good cases, remembering X plus some additional algebraic data (for example the structure sheaf), one can recover X. This point of view has proved extremely fruitful as we shall see. In any case, it often happens that there are resolutions of X, the *crepant resolutions*, for which the quantum theory is the same as that for X. We will come back to this later.

There is, of course, a far more classical interpretation of the Euler characteristic, the topological interpretation. The classical interpretation of the Euler characteristic in terms of triangulations tells us that the Euler characteristic is the alternating sum of the Betti numbers, namely, the ranks of the cohomologies of the space in question. Thus, a natural question is whether there is a cohomology theory for an orbifold that is physical and that simultaneously produces the appropriate Euler characteristic (2.0.1). One is first tempted to consider equivariant cohomology $H^*_G(M) = H^*(M \times_G EG)$ but unfortunately the relation between cohomology and Euler characteristic breaks down, for the expression (2.0.1) is not recovered.

Considering the orbifold $\mathsf{X} = [*/G]$ consisting of a finite group acting on a single point gives us a clue into the right answer. In this case, $\chi_{\mathbf{Orb}}([*/G])$ becomes the number of pairs of commuting elements in G divided by $|G|$. An amusing exercise in finite group theory readily shows that this is the same as the number of conjugacy classes of elements in G. Given a finite group there are two basic quantities that we can consider, its group cohomology $H^*(BG)$ and its representation ring $R(G)$. While equivariant cohomology is akin to group cohomology, it is *equivariant K-theory* $K_G(M)$ that is intimately related to representation theory. For a start, $K_G(*) = R(G)$.

As a first test, we consider an orbifold $\mathsf{X} = [M/G] \cong [N/H]$ and see whether the theory is invariant under the representation. This is not too hard (see for example [**29, 1**]), and hence it fully deserves the name of *orbifold K-theory* and can unambiguously be written as $K_{\mathbf{Orb}}(\mathsf{X}) = K_G(M) \cong K_H(M)$.

The second test is to see whether we can recover Formula (2.0.1). That this is possible was first observed by Atiyah and Segal [**6**]. The idea is to use the Segal character of an equivariant vector bundle. Let us remember that the basic cocycles of equivariant K-theory are G-equivariant vector bundles [**4**], namely bundles $p\colon E \to M$ over the G-manifold M with a G-action by bundle automorphisms on all of E that extends the action on M (considered as the zero section) and that is fiberwise linear. Should there be a fixed point $m \in M$, then $E_m := p^{-1}(m)$ becomes a representation of G; in particular, if the space M is a point then a G-equivariant vector bundle over M is the same as a representation of G (by choosing a basis we get a matrix for every $g \in G$).

The (Segal) character of an equivariant vector bundle is an *isomorphism* [**36, 34**] of the form

$$K_G(M) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{(g)} K(M^g)^{C(g)} \otimes \mathbb{C}, \tag{2.0.2}$$

where the sum is over all conjugacy classes (g) of elements $g \in G$.

The character isomorphism is explicitly given by the expression

$$\begin{array}{rcl} K_G(M)\otimes\mathbb{C} & \to & K(M^g)^{C(g)}\otimes\mathbb{C} \\ E\otimes 1 & \mapsto & \mathrm{char}(E)(g)=\sum_\zeta (E|_{M^g})_\zeta\otimes\zeta. \end{array}$$

Here the sum is over all roots of unity ζ, the symbol $(\ )_\zeta$ denotes the ζ-eigenspace of g, and finally M^g is the subspace of fixed point under g of M. We call this isomorphism the *Segal localization formula* (for it localizes equivariant K-theory to ordinary K-theory of the fixed point sets).[3] Clearly, in the case in which M is a point, this recovers the usual theory of characters for the finite-dimensional representations of a finite group.

From Segal's isomorphism (2.0.2) we conclude immediately that [**6, 9, 41**]

$$\mathrm{rank}K^0_G(M)-\mathrm{rank}K^1_G(M)=\sum_{(g)}\chi(M^g/C(g))=\frac{1}{|G|}\sum_{gh=hg}\chi(M^{g,h})=\chi_{\mathbf{Orb}}(\mathsf{X}).$$

Here we have applied the algebraic equality

$$\chi_{\mathbf{Orb}}(\mathsf{X})=\sum_{(g)}\chi(M^g/C(g)),$$

which follows by an inclusion-exclusion argument [**22**]; in the next section we talk about a geometric explanation for this algebraic fact.

For now let us mention that the theory described in this section can be generalized to orbifolds that are not necessarily global quotients [**29, 1**]. This is done as follows. We will denote by X_0 and X_1 the set of objects and morphism of our groupoid respectively, and the structure maps by

$$\mathsf{X}_1\ {}_t\!\times_s \mathsf{X}_1 \xrightarrow{m} \mathsf{X}_1 \xrightarrow{i} \mathsf{X}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathsf{X}_0 \xrightarrow{e} \mathsf{X}_1,$$

where $\mathsf{X}_1\ {}_t\!\times_s \mathsf{X}_1$ is the subspace of $\mathsf{X}_1\times\mathsf{X}_1$ such that whenever $(\alpha,\beta)\in\mathsf{X}_1\ {}_t\!\times_s \mathsf{X}_1$ then the target of α equals the source β; s and t are the source and the target maps on morphisms, m is the composition arrows, i gives us the inverse morphism, and e assigns the identity arrow to every object.

We define a *vector orbibundle* over X to be a pair (E,τ) where E is an ordinary vector bundle over X_0 and $\tau\colon s^*E\xrightarrow{\cong}t^*E$ is an isomorphism of vector bundles over X_1.

The set of isomorphism classes of such orbibundles is denoted by $\mathit{Orbvect}(\mathsf{X})$ and its Grothendieck group by $K^0_{\mathrm{orb}}(\mathsf{X})$ [**29**].

This coincides with equivariant K-theory if the orbifold happens to be of the form $[M/G]$.

3. Loop Orbifolds

Let us consider now a Riemannian metric on M. There is then a family of canonically defined operators: the Laplacians on k-forms Δ^k. These are related

[3] Remarkably enough this is indeed related to the localization of equivariant K-theory as an $R(G)$-module with respect to prime ideals [**38**].

to a quantum field theory whose fields are maps from intervals the circle to M. Roughly speaking, the Lagrangian of the theory is given by

$$L(\phi) = \frac{1}{2}\int |d\phi|^2.$$

All the information of such quantum theory is contained in the spectrum of the Laplacian. Recovering the classical theory from the quantum one is "hearing the shape of the drum." In any case, the Feynman functional integration approach for the theory allows us to compute an integral over the free loop space of the manifold $\mathcal{L}(M) = \mathrm{Map}(S^1; M)$ by stationary phase approximation as an integral over M.

This quantum field formalism is related to the heat equation

$$\partial_t \omega + \Delta^k \omega = 0,$$

whose solution is given by the heat flow $e^{-t\Delta^k}$. In particular the fundamental solution for the trace of the heat kernels is given by

$$\sum (-1)^k \mathrm{Tr}(e^{-t\Delta^k}) = \int_{\mathcal{L}M} e^{t^{-1}L(\phi)}\mathcal{D}\phi,$$

where $\mathcal{D}\phi$ is the formal part of the Wiener measure on $\mathcal{L}M$.

It turns out that the the sum of the traces of the heat kernels is independent of t. The long time limit of this sum equals the Euler characteristic (by recalling Hodge's theorem, which identifies the k-th Betti number of M as the dimension of the kernel of Δ^k), and the short time behaviour is given by an integral of a complicated curvature expression.

If the dimension of the manifold is 2, this equality of long and short time behaviour of the heat flow leads to the Gauss-Bonet theorem

$$\int_M K dA = \chi(M), \tag{3.0.3}$$

where K is the Gaussian curvature and dA is the volume element.

In fact we have oversimplified: we can do better than to simply recover the Euler characteristic. Suppose that M is a spin manifold; then we can recover by this procedure the *index of the Dirac operator*. We will come back to this observation in the next section. But before we do that let us see how we stand in the orbifold case.

To try to apply these methods to an orbifold X (replacing the rôle of M above), we must be able to say what is the candidate to replace $\mathcal{L}M$. This was done for a general orbifold in [**28**].

The idea is that to a groupoid X we must assign a new (infinite-dimensional) groupoid LX that takes the place of the free loopspace of M. The construction in [**28**] furthermore commutes with the functor B from groupoids to spaces defined in the first section, as shown in [**31**], in the sense that there is an homotopy equivalence

$$B\mathsf{LX} \simeq \mathcal{L}B\mathsf{X}.$$

In the case in which $\mathsf{X} = [M/G]$, we proved that LX admits a very concrete model defined as follows.

The objects of the loop groupoid are given by

$$(\mathsf{LX})_0 := \bigsqcup_{g\in G} \mathcal{P}_g,$$

where $\mathcal{P}_g$ is the set of all pairs (γ, g) with $\gamma : \mathbb{R} \to X$ and $g \in G$ with $\gamma(t)g = \gamma(2\pi + t)$.

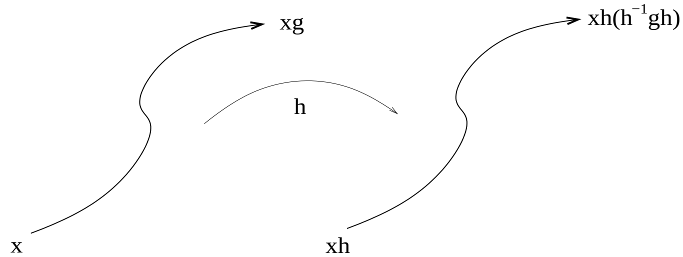
The space of arrows of the loop groupoid is

$$(\mathsf{LX})_1 := \bigsqcup_{g \in G} \mathcal{P}_g \times G,$$

and the action of G in $\mathcal{P}_g$ is by translation in the first coordinate and conjugation in the second; that is, a typical arrow in the loop groupoid looks like

$$(\gamma, g) \xrightarrow{((\gamma,g);h)} (\gamma \cdot h, h^{-1}gh),$$

or pictorially:



We are ready to state the basic localization principle.

Theorem 3.0.1 (The Localization Principle [**28**]). *Let* X *be an orbifold and* LX *its loop orbifold. Then the fixed orbifold under the natural circle action by rotation of loops is*

$$(\mathsf{LX})^{S^1} = I(\mathsf{X}), \tag{3.0.4}$$

where the groupoid $I(\mathsf{X})$ *has as its space of objects*

$$I(\mathsf{X})_0 = \{\alpha \in \mathsf{X}_1 \colon s(\alpha) = t(\alpha)\} = \coprod_{m \in \mathsf{X}_0} \mathrm{Aut}_{\mathsf{X}}(m)$$

and its space of arrows is

$$I(\mathsf{X})_1 = Z(I(\mathsf{X}_0)) = \{g \in \mathsf{X}_1 \colon \alpha \in I(\mathsf{X})_0 \Rightarrow g^{-1}\alpha g \in I(\mathsf{X})_0\};$$

a typical arrow in $I(\mathsf{X})$ *from* α_0 *to* α_1 *looks like*

$$\alpha_0 \circlearrowleft \circ \underset{g^{-1}}{\overset{g}{\rightleftarrows}} \circ. \circlearrowright \alpha_1^{-1}$$

While for a smooth manifold we have

$$M = (\mathcal{L}M)^{S^1},$$

we have, by contrast,

$$\mathsf{X} \subset I(\mathsf{X}) = (\mathsf{LX})^{S^1},$$

; so we expect the Euler characteristic, K-theory, and so on to localize in $I(\mathsf{X})$ rather than in X. The orbifold $I(\mathsf{X})$ is called in the mathematical literature the *inertia orbifold* of X, and it is, as Chen and Ruan [**12**] have pointed out (and as is reflected in their terminology), the classical geometrical manifestation of the *twisted sectors* of orbifold string theory [**17**].

Indeed, we have that for a general orbifold

$$\chi_{\mathbf{Orb}}(\mathsf{X}) = \chi(I(\mathsf{X}))$$

and

$$K^*_{\mathrm{orb}}(\mathsf{X}) \otimes \mathbb{C} \cong K^*(I(\mathsf{X})) \otimes \mathbb{C}.$$

For example, in the case of a global quotient $\mathsf{X} = [M/G]$, one can readily verify that

$$I(\mathsf{X}) = \coprod_{(g)} [M^g/C(g)], \tag{3.0.5}$$

recovering thus Segal's localization formula and the orbifold Euler characteristic of the previous section.

In [**30**], the *ghost loop space* $\mathcal{L}_s B\mathsf{X}$ is defined as the subspace of elements $\gamma \in \mathcal{L}B\mathsf{X}$ so that the composition with the canonical projection $\pi_{\mathsf{X}}\colon B\mathsf{X} \to X$, $\pi_{\mathsf{X}} \circ \gamma$ is constant. That paper proves the homotopy equivalence

$$BI(\mathsf{X}) \simeq \mathcal{L}_s B\mathsf{X}.$$

4. The McKay Correspondence

Let us consider now a classical example. Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$; then $X = \mathbb{C}^2/G$ is called a *Kleinian quotient singularity*; see [**40, 14**] for more details and historical discussion. In the second half of the 19th century, Klein classified the possible groups G as either cyclic, dihedral or binary dihedral and gave equations for these singularities in $\mathbb{C}^3$. Let us consider the simplest case in which $G \cong \mathbb{Z}/r\mathbb{Z}$. We can realize X as a subvariety of $\mathbb{C}^3$ by

$$X\colon z^r = xy$$

or, in parametric form,

$$\begin{aligned} x &= u^r \\ y &= v^r \\ z &= uv \end{aligned} \tag{4.0.6}$$

as the image of a map $\mathbb{C}^2 \to \mathbb{C}^3$ by G-invariant polynomials. We can resolve the singularity very easily in this case by taking $(r-1)$-blow ups to obtain

$$Y \xrightarrow{\phi} X$$

where the exceptional divisor is

$$\phi^{-1}(0) = E_1 \cup E_2 \cup \cdots \cup E_{r-1}$$

whose incidence graph is A_{r-1}.

On the other hand, G clearly has $r-1$ nontrivial irreducible representations.

The *McKay correspondence* establishes (among other things) a one-to-one correspondence between the number of components of the exceptional divisor in a minimal resolution of the singularity and the number of nontrivial irreducible representations of G. Notice that in our example this is equivalent to the statement that *the orbifold Euler characteristic of* X *is the same as the ordinary Euler characteristic of* Y. So one may expect that some functional integral argument may be provided to prove the McKay correspondence.

There is in fact a rigorous version of the functional integration method in algebraic geometry discovered by Kontsevich [**24**] and known as *motivic integration*. We now briefly outline the construction of this method.

Given a smooth complex variety Y, one can define its *arc space* JY. This is a scheme whose $\mathbb{C}$-points are arcs $\gamma\colon \mathrm{Spec}(\mathbb{C}[[t]]) \to Y$. The scheme JY is

obtained as the inverse limit of the *jet schemes* J_mY, whose $\mathbb{C}$-points are jets $\gamma_m\colon \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to Y$. The morphisms $J_pY \to J_mY$, for $0 \le m \le p \le \infty$, are given by truncation. For any effective divisor $D \subset Y$, one can define an order function

$$\mathrm{ord}_D : JY \to \mathbb{Z}_{\ge 0} \cup \{\infty\},$$

which to each arc γ associates its order of contact $\mathrm{ord}_D(\gamma)$ along D. The idea is then to "integrate these functions," in some reasonable sense. But first one needs to introduce the algebra of measurable sets and the measure. The first is easily defined as the algebra generated by *cylinder sets* in JY, namely, inverse images of constructible sets on finite levels J_mY. The measure will then take values in the so-called *motivic ring*.

The motivic ring is constructed as follows: we fix a complex variety X and assume that Y is an X-variety (that is, a complex algebraic variety of finite type over X). Let $K_0(\mathcal{V}ar_X)$ be the ring generated by X-isomorphism classes of X-varieties subjected to the relation

$$\{V\} = \{V \setminus W\} + \{W\}$$

whenever W is a closed variety of a X-variety V. The product is defined by

$$\{V\} \cdot \{W\} = \{V \times_X W\}.$$

The zero of this ring is $\{\varnothing\}$, and the identity is $\{X\}$. We let

$$\mathcal{M}_X = K_0(\mathcal{V}ar_X)[\mathbf{L}_X^{-1}],$$

where $\mathbf{L}_X$ is the class of the affine line over X. Finally the motivic ring is the completion $\hat{\mathcal{M}}_X$ of $\mathcal{M}_X$ under a certain natural dimension filtration [**26, 15, 13**].

Via composition, every subvariety of the jet schemes of Y can be viewed as an X-variety. Thus, one can define the *motivic measure* of a cylinder $C \subseteq JY$ by fixing a large enough integer m such that C is the inverse image of a constructible set $C_m \subseteq J_mY$ and then setting

$$\mu(C) = \{C_m\} \cdot \mathbf{L}_X^{-m \dim Y} \in \hat{\mathcal{M}}_X.$$

Then, by suitable stratification, one defines the *motivic integral*

$$\int_{JY} \mathbf{L}_X^{-\mathrm{ord}_D} d\mu \in \hat{\mathcal{M}}_X.$$

For instance, if $D = \sum a_j D_j$ is a simple normal crossing divisor and we define

$$D_J^\circ = \bigcap_{j \in J} D_j \setminus \bigcup_{i \notin J} D_i,$$

then one has

$$\int_{JY} \mathbf{L}_X^{-\mathrm{ord}_D} d\mu = \sum_{J \subseteq I} \{D_J^\circ\} \prod_{j \in J} \frac{\mathbf{L}_X - 1}{\mathbf{L}_X^{a_j+1} - 1}.$$

The power of this theory is a *change of variable formula*; this allows us to reduce to computing integrals for divisors with simple normal crossings (hence apply the above formula) by replacing any effective divisor D on Y by $D' = K_{Y'/Y} + g^*D$, where $g : Y' \to Y$ is a simple normal crossing resolution of the pair (Y, D). The theory can be also extended to singular varieties (under suitable conditions): in this case the measure itself needs to be opportunely "twisted" to make the change of variable formula work. The resulting measure is called *Gorenstein measure* and denoted by μ^{Gor}.

We can now review the *motivic McKay correspondence* [**15, 26, 35**]. To give a formulation of this correspondence that better fits in the "localization" context of this paper, we need to further quotient the ring $K_0(\mathcal{V}ar_X)$ by identifying X-varieties that become isomorphic after some étale base change $X'_k \to X_k \subseteq X$ of each piece X_k of a suitable stratification $X = \bigsqcup X_k$ of X. We obtain in this way a new ring: $K_0(\mathcal{V}ar_X)^{et}$. This leads to the definition of a different motivic ring, which we denote by $\hat{\mathcal{M}}_X^{et}$ (the reader will notice that, if X is a point, then we are not changing anything).

Let $\mathsf{X} = [M/G]$, where M is a quasiprojective variety and G is a finite group, let $X = M/G$, and assume that X is Gorenstein. We can find a resolution of singularities $Y \to X$ with relative canonical divisor $K_{Y/X}$ having simple normal crossings. Write $K_{Y/X} = \sum a_j D_j$. Then the McKay correspondence is given by the identity

$$\sum_{J\subseteq I}\{D_J^\circ\}\prod_{j\in J}\frac{\mathbf{L}_X-1}{\mathbf{L}_X^{a_j+1}-1} = \sum_{(g)}\{M^g/C(g)\}\mathbf{L}_X^{w(g)} \quad \text{in } \hat{\mathcal{M}}_X^{et}, \tag{4.0.7}$$

where the sum in the left side runs over conjugacy classes (g) in G and $w(g)$ are integers depending on the local action of g on the normal bundle of M^g in M.

For instance, by noticing that the Euler characteristic defines a ring homomorphism

$$\chi\colon K_0(\mathcal{V}ar_X)^{et} \to \mathbb{Z},$$

it is easy to see that Formula (4.0.7) implies the classical McKay correspondence,[4] which in particular says that the orbifold Euler characteristic is equal to the ordinary Euler characteristic of the resolution if the latter is crepant.

The proof of Formula (4.0.7) breaks into three parts. By the change of variable formula, one has

$$\int_{JX}\mathbf{L}_X^0 d\mu^{Gor} = \sum_{J\subseteq I}\{D_J^\circ\}\prod_{j\in J}\frac{\mathbf{L}-1}{\mathbf{L}^{a_j+1}-1} \quad \text{in } \hat{\mathcal{M}}_X.$$

Then, by an accurate study of lifts of the arcs of X to arcs on M, one proves that

$$\int_{JX}\mathbf{L}_X^0 d\mu^{Gor} = \sum_{(H)}\{X^H\}\sum_{(h)}\mathbf{L}_X^{w(h)} \quad \text{in } \hat{\mathcal{M}}_X.$$

Here the first sum runs over conjugacy classes (H) of subgroups of G, $X^H \subseteq X$ is the image of the set of points on M whose stabilizer is H, and the last sum is taken over conjugacy classes in H. The above identity is the core of the proof. Finally, one shows that

$$\sum_{(H)}\{X^H\}\sum_{(h)}\mathbf{L}_X^{w(h)} = \sum_{(g)}\{M^g/C(g)\}\mathbf{L}_X^{w(g)} \quad \text{in } \hat{\mathcal{M}}_X^{et}.$$

Here is where we need to pass to the ring $\hat{\mathcal{M}}_X^{et}$. This last part can be easily verified using certain properties of Deligne-Mumford stacks (see [**13**]). In general, if we do not perform the additional localization in the relative motivic ring, but instead work with the ring $\hat{\mathcal{M}}_X$, we do not expect the last identity to hold.

[4]Here we are referring only to the counting statement, and not that we recover the full incidence graph of $\phi^{-1}(0)$ from the representation theory of G, as the classical correspondence establishes.

These results have been extended to general (not necessarily global quotient) orbifolds independently by Yasuda [**43**] and by Lupercio-Poddar [**27**].

In [**13**], we used a natural homomorphism from $K_0(\mathcal{V}ar_X)$ to the ring of constructible functions $F(X)$ on X to associate to any motivic integral an element in $F(X)_{\mathbb{Q}}$, that is, a rational-valued constructible function on X. In fact, one observes that this construction factors through $K_0(\mathcal{V}ar_X)^{et}$.[5] The result is the following localization formula for constructible functions:

$$\sum_{J\subseteq I} \frac{(f|_{D_J^\circ})_* \mathbf{1}_{D_J^\circ}}{\prod_{j\in J}(a_j+1)} = \sum_{(g)} (\pi_g)_* \mathbf{1}_{M^g/C(g)} \quad \text{in } F(X), \tag{4.0.8}$$

where $\pi_g : M^g/C(g) \to X$ is the morphism naturally induced by the quotient map $\pi : M \to X$ [**13**, Theorem 6.1].

Motivic integration was used in [**13**] to define the *stringy Chern class* $c_{\mathrm{str}}(X)$ of X. In the case at hand, we use the MacPherson transformation [**33**] to deduce from (4.0.8) the following localization formula for the stringy Chern class of a quotient [**13**, Theorem 6.3]:

$$c_{\mathrm{str}}(X) = \sum_{(g)} (\pi_g)_* c_{\mathrm{SM}}(M^g/C(g)) \quad \text{in } A_*(X),$$

where $c_{\mathrm{SM}}(M^g/C(g))$ is the Chern-Schwartz-MacPherson class of $M^g/C(g)$ [**33**]. This generalizes and implies Batyrev's formula for the Euler characteristic [**8**].

5. Index Theory

Now the situation is as follows. Suppose that we have a compact symplectic $2m$-dimensional manifold N with symplectic form ω and that $H\colon N \to \mathbb{R}$ is the Hamiltonian function of a Hamiltonian circle action. Let F_α be the critical manifolds of H (namely the fixed points of the action) with critical values H_α. The Liouville volume form on N is $\omega^m/m!$. The Duistermaat-Heckman formula reads [**7, 19**]

$$\int_N e^{\hbar H} \frac{\omega^m}{m!} = \sum_\alpha e^{\hbar H_\alpha} \int_{F_\alpha} \frac{e^\omega}{E_\alpha},$$

where E_α is the equivariant Euler class of the normal bundle of F_α in N. If $\hbar$ is taken as purely imaginary, the integral over N is oscillatory, the submanifolds F_α are the stationary points of H, and the right-hand side of this formula is given by stationary phase approximation.

Witten [**5**] had the idea of using the Duistermaat-Heckman formula in the case $N = \mathcal{L}M$, the free loop space of a manifold M, with Hamiltonian

$$H(\gamma) = \frac{1}{2}\oint_{S^1} |\gamma'(t)|^2 dt.$$

In this case Atiyah defines a symplectic form on $\mathcal{L}M$ whenever M is compact and orientable. Then he goes on to show that *when M is a Spin manifold, $\mathcal{L}M$ is orientable*. Moreover, he shows that the left-hand side of the corresponding Duistermaat-Heckman formula is the heat kernel expression for the index of the Dirac operator while the right-hand side is the $\hat{A}$-genus, thus giving the Atiyah-Singer index theorem.

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[5] In particular, this tells us that the identification performed to define $\hat{\mathcal{M}}_X^{et}$ does not trivialize the ring too much, as we can still recover all the information in $F(X)$.

We do the same now for the loop groupoid. In order to simplify the calculation, we will consider the case of a global quotient $\mathsf{X} = [M/G]$, but everything that we will say generalizes to general (non–global-quotient) orbifolds. We will suppose thus that M is a compact, even-dimensional spin manifold such that for every $g \in G$ the map $g : M \to M$ given by the action is a spin-structure-preserving isometry. We will argue that applying stationary phase approximation to the integral[6]

$$\int_{\mathcal{P}_g} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}\mathcal{D}\phi \tag{5.0.9}$$

one obtains

$$Spin(M,g) := ind_g(D^+) = \mathrm{tr}(g|_{kerD^+}) - \mathrm{tr}(g|_{cokerD^+}),$$

the value of the g-index of the Dirac operator D^+ over M. Here E is the energy of the path (Hamiltonian)

$$E(\phi) := \frac{1}{2}\int_0^{2\pi} |\phi'(t)|^2 dt,$$

$\mathcal{D}\phi$ denotes the formal part of the Weiner measure on $\mathcal{P}_g$, T_ϕ is the tangent space at $\phi \in \mathcal{P}_g$, and S^+, S^- denote the two half-spin representations of $Spin(2m)$ ($2m = \dim M$).

The real numbers act on $\mathcal{P}_g$ by shifting the path

$$\begin{array}{rcl} \mathcal{P}_g \times \mathbb{R} & \to & \mathcal{P}_g \\ (f,s) & \mapsto & f_s : \mathbb{R} \to M \\ & & f_s(t) := f(t-s) \end{array}$$

and the fixed point set of this action on $\mathcal{P}_g$ consists of the constant maps to M^g (the fixed point set of the action of g in M), that is,

$$(\mathcal{P}_g)^{\mathbb{R}} \cong M^g.$$

Applying the stationary phase approximation (see [**5**, Formula 2.2]) to the integral (5.0.9), we get

$$\int_{(\mathcal{P}_g)^{\mathbb{R}}} \frac{e^{-tE(\phi)}}{\prod_j (tm_j - i\alpha_j)} = \int_{M^g} \frac{1}{\prod_j (tm_j - i\alpha_j)}, \tag{5.0.10}$$

where the energy of the constant paths is zero, the m_j are rotation numbers normal to M^g, and the α_j are the Chern roots, so that the total Chern class of the normal bundle N to M^g is given by

$$\prod_j (1 + \alpha_j).$$

5.1. Normal bundle. For $f \in \mathcal{P}_g$, the tangent space T_f at f can be seen as the space of maps

$$\sigma : \mathbb{R} \to f^*TM$$

such that $\sigma(t)dg_{\sigma(t)} = \sigma(2\pi + t)$, so for the constant map at $x \in M^g$, its tangent space is equal to the space of maps

$$\sigma : \mathbb{R} \to T_xM$$

with $\sigma(t)dg_x = \sigma(2\pi + t)$.

[6]In [**5**] it is explained how to make sense of this integral.

We can split the vector space T_xM into subspaces $N(\theta)$ that consist of 2-dimensional spaces on which dg_x rotates every vector by θ (see [**25**]):

$$T_xM \cong N(0) \oplus \bigoplus_\theta N(\theta).$$

It is clear that the number of θ is finite, that we could choose them in the interval[7] $0 < \theta < \pi$, and that $N(0) \cong T_xM^g$.

The constant functions

$$\{\sigma : \mathbb{R} \to T_xM^g \cong N(0) \mid \ \sigma \text{ is constant}\} \subset T_x\mathcal{P}_g$$

give the directions along M^g. We are interested in finding a description of the normal directions of M^g in $T_x\mathcal{P}_g$.

Let $2s(\theta) := \dim_{\mathbb{R}} N(\theta)$ and, for $l = 1, \ldots, s(\theta)$, let $N_l(\theta)$ be the 2-dimensional subspaces fixed by dg_x through the rotation of θ. Then any $\sigma \in T_x\mathcal{P}_g$ can be seen as

$$\sigma = \sum_{l,\theta} \sigma_l^\theta \qquad \text{with} \qquad \sigma_l^\theta : \mathbb{R} \to N_l(\theta).$$

Let $N_l(\theta)^{\mathbb{C}}$ be the complexification $N_l(\theta) \otimes \mathbb{C}$. Then

$$N_l(\theta)^{\mathbb{C}} \cong L_l \oplus \overline{L_l},$$

where L_l is a complex line bundle, the action of dg_x on L_l is by multiplication by $e^{i\theta}$, and $\overline{L_l}$ is the conjugate bundle of L_l (see [**25**, p. 226]). The map

$$\sigma_l^\theta : \mathbb{R} \to N_l(\theta) \subset N_l(\theta)^{\mathbb{C}}$$

can be seen in $L_l \oplus \overline{L_l}$ via a Fourier expansion as

$$\sigma_l^\theta(t) = \sum_{k \in \mathbb{Z}} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \begin{pmatrix} e^{itk} e^{it\frac{\theta}{2\pi}} & 0 \\ 0 & e^{itk} e^{-it\frac{\theta}{2\pi}} \end{pmatrix} \tag{5.1.1}$$

with $a_k \in L_l$, $b_k \in \overline{L_l}$, $a_k = \overline{a_{-k}}$ and $b_k = \overline{b_{-k}}$ (the last two equations hold because $\sum_k a_k e^{itk}$ and $\sum_k b_k e^{itk}$ are real for all t; in particular $a_0, b_0 \in \mathbb{R}$).

Then the tangent bundle to $T_x\mathcal{P}_g$ can be decomposed as an infinite direct sum

$$T_xM \oplus (T_xM^{\mathbb{C}})_1 \oplus (T_xM^{\mathbb{C}})_2 \oplus \cdots$$

with

$$(T_xM^{\mathbb{C}})_n \cong (N(0)^{\mathbb{C}})_n \oplus \bigoplus_\theta (N(\theta)^{\mathbb{C}})_n$$

where the circle acts in each $(N(\theta)^{\mathbb{C}})_n$ by rotation number n. The coefficients (a_k, b_k) of the Fourier expansion of (5.1.1) take values in $(N(\theta)^{\mathbb{C}})_k$ for $k > 0$, $(a_0, b_0) \in N(\theta)$, and $(a_k, b_k) = (\overline{a_{-k}}, \overline{b_{-k}})$ for $k < 0$.

As $T_xM \cong N(0) \oplus \bigoplus_\theta N(\theta)$ and $N(0) \cong T_xM^g$ represent the directions along M^g, the normal bundle to M^g in $\mathcal{P}_g$ can be represented as

$$\left\{(N(0)^{\mathbb{C}})_1 \oplus (N(0)^{\mathbb{C}})_2 \oplus \cdots\right\} \oplus \bigoplus_\theta \left\{N(\theta) \oplus (N(\theta)^{\mathbb{C}})_1 \oplus (N(\theta)^{\mathbb{C}})_2 \oplus \cdots\right\}.$$

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[7] For simplicity we will assume that the eigenvalue π is not included, in order to avoid the use of Pontrjagin classes. The result still holds with π as rotation number.

Let the Chern class of $N(\theta)$ be

$$\prod_{k=1}^{s(\theta)}(1+y_k^\theta),$$

so its g-Chern character is

$$ch_g(N(\theta)) = \sum_{k=1}^{s(\theta)} ch(N_k(\theta))\chi(g) = \sum_{k=1}^{s(\theta)} e^{y_k^\theta + i\theta};$$

then the g-Chern class of the complexification of $N(\theta)$ is

$$\prod_{k=1}^{s(\theta)}(1+y_k^\theta+i\theta)(1-y_k^\theta-i\theta).$$

If we let x_k denote the Chern classes of M^g, then the denominator in (5.0.10) with $t=1$ becomes

$$\prod_{j=1}^{s(0)}\prod_{p=1}^{\infty}\left(p^2+x_j^2\right)\prod_\theta\left\{\prod_{k=1}^{s(\theta)}(y_k^\theta+i\theta)\prod_{p=1}^{\infty}\left(p^2+(y_k^\theta+i\theta)\right)\right\},$$

which is formally

$$\prod_{j=1}^{s(0)}\left(\prod_{p=1}^{\infty}p^2\right)\frac{\sinh(\pi x_j)}{\pi x_j}\prod_\theta\left\{\prod_{k=1}^{s(\theta)}\left(\prod_{p=1}^{\infty}p^2\right)(y_k^\theta+i\theta)\frac{\sinh(\pi(y_k^\theta+i\theta))}{\pi(y_k^\theta+i\theta)}\right\}.$$

Replacing the infinite product of the p^2 by its renormalized factor 2π, we get

$$\prod_j\frac{2\sinh(\pi x_j)}{x_j}\prod_\theta\left\{\prod_k 2\sinh(\pi(y_k^\theta+i\theta))\right\},$$

which is the same as

$$\prod_j\frac{\sinh(x_j/2)}{x_j/2}\prod_\theta\left\{\prod_k\frac{\sinh((y_k^\theta+i\theta)/2)}{1/2}\right\} \tag{5.1.2}$$

provided we interpret $\prod_{p=1}^\infty t$ as $t^{\zeta(0)}$ where $\zeta(s)$ is the Riemann zeta function. As $\zeta(0)=-\frac{1}{2}$, in each component we get a factor of t which cancels with the factor t^{-1} that arises from replacing x_j by x_j/t and $y_k^\theta+i\theta$ by $(y_k^\theta+i\theta)/t$. Our use of the stationary phase approximation is independent of t, and setting $t=2\pi$ we get formula (5.1.2).

In the notation of [**25**, p. 267] formula (5.1.2) is equivalent to

$$\left(\hat{A}(M^g)\prod_\theta\hat{A}(N(\theta))\right)^{-1},$$

which after replacing it in the denominator of (5.0.10) and integrating over M^g matches the formula for $Spin(M,g)$ [**25**, Th. 14.11]:

$$Spin(M,g) = (-1)^{\tau_g}\hat{A}(M^g)\left\{\prod_\theta\hat{A}(N(\theta))\right\}[M^g].$$

We conclude that after applying the stationary phase approximation to (5.0.9), we obtain the g-index of the Dirac operator.

Proposition 5.1.1. *The path integral*

$$\int_{\mathcal{P}_g} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}d\phi = Spin(M,g)$$

equals $\mathrm{ind}_g(D^+)$*, the* g*-index of the Dirac operator over* M*.*

5.2. The G-index and Kawasaki's formula. The G-index of the Dirac operator is an element of $R(G)$, the representation ring of G. Using localization, its dimension is equal to

$$\mathrm{ind}_G(D^+) = \frac{1}{|G|}\sum_{g\in G}\mathrm{ind}_g(D^+) = \frac{1}{|G|}\sum_{g\in G} Spin(M,g).$$

But instead of summing over all the elements g in G, we could sum over the conjugacy classes of G. It is clear that $Spin(M,g) = Spin(M, h^{-1}gh)$. The size of the conjugacy class (g) of g is $\frac{|G|}{|C(g)|}$ where $C(g)$ is the centralizer of G, that is, the set of elements which commute with g (equivalently, the fixed point set of the action of G in g via conjugation). Thus, we obtain

$$\mathrm{ind}_G(D^+) = \sum_{(g)}\frac{1}{|C(g)|}Spin(M,g).$$

We would like to derive a formula that depends on the twisted sectors (inertia groupoid) of the orbifold $\mathsf{X} = [M/G]$, and this clearly matches our previous description. In [**28**] it was argued that the fixed point set of the action of $\mathbb{R}$ in the loop groupoid LX was precisely $I(\mathsf{X})$ the inertia groupoid of X; then, applying stationary phase approximation to

$$\int_{\mathsf{LX}} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}\mathcal{D}\phi,$$

which can be rewritten as

$$\sum_{(g)}\frac{1}{|C(g)|}\int_{\mathcal{P}_g} e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}\mathcal{D}\phi,$$

we get the G-index of the Dirac operator,

$$\mathrm{ind}_G(D^+) = \sum_{(g)}\frac{(-1)^{\tau_g}}{|C(g)|}\int_{M^g}\hat{A}(M^g)\prod_\theta \hat{A}(N(\theta)_g).$$

Which can be shown to coincide with the formula given by Kawasaki [**23**, p. 139] for the index theorem for V-manifolds. Thus, the localization principle applies in this case.

6. The Elliptic Genus

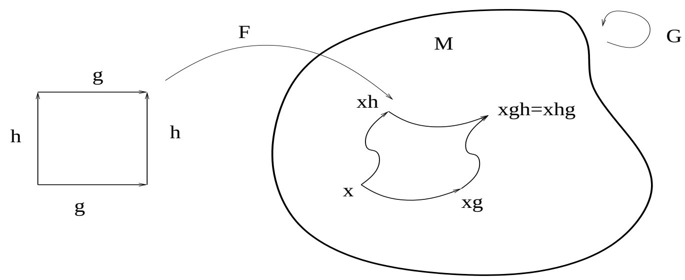
We move on now to localizing functional integrals in the double loopspace $\mathcal{L}^2M = \mathcal{L}\mathcal{L}M = \mathrm{Map}(T, \mathrm{Map}(T,M)) = \mathrm{Map}(T^2, M)$, where $T = S^1$ and T^2 is the 2-torus. By performing the corresponding functional integral over $\mathcal{L}^2M$, we should obtain the index of the dirac operator over $\mathcal{L}M$ considered by Witten in [**42**] and known as the elliptic genus [**39**]. This has been verified by Ando and Morava [**3**]. We want to perform the calculation in the orbifold case (cf. [**2**]).

Let the groupoid X be $[M/G]$, and let the torus T be represented by the groupoid $[\mathbb{R}^2/\mathbb{Z}\oplus\mathbb{Z}]$. The *double loop groupoid* $\mathsf{L}^2\mathsf{X}$ is the category with smooth functors $\mathsf{T}\to\mathsf{X}$ as objects and natural transformations between functors as morphisms.

A morphism in $\mathsf{L}^2\mathsf{X}$ can be seen as

$$\begin{array}{ccc} \mathbb{R}^2\times(\mathbb{Z}\oplus\mathbb{Z}) & \longrightarrow & M\times G \\ \downdownarrows & & \downdownarrows \\ \mathbb{R}^2 & \longrightarrow & M, \end{array}$$

that is, as a map $F\colon\mathbb{R}^2\to M$ together with a homomorphism $H\colon\mathbb{Z}\oplus\mathbb{Z}\to G$ such that F is equivariant with respect to H. This is equivalent to choosing a pair of commuting elements $g,h\in G$ such that $F(1,0)=F(0,0)g$, $F(0,1)=F(0,0)h$ and in general $F(n,m)=F(0,0)g^nh^m$.



The group $\mathbb{R}^2$ acts naturally by translations on the double loop groupoid. This action factors through $\mathbb{R}^2/\{|G|\mathbb{Z}\oplus|G|\mathbb{Z}\}$ because every orbifold loop can be closed in M/G.

The fixed points under the action of $\mathbb{R}^2$ are the constant double loops; they are uniquely determined by a choice of a point in M and two commuting elements in G.

The groupoid of *ghost double loops* is the groupoid whose objects is the set of functors

$$Funct([*/\mathbb{Z}\oplus\mathbb{Z}],[M/G])$$

and whose morphisms are natural transformations (i.e., it is a groupoid $[(Funct([*/\mathbb{Z}\oplus\mathbb{Z}],[M/G]))/G]$ with G acting by conjugation on the functors).

Here we will apply the stationary phase approximation formula to the double loop groupoid, which we have shown above to be endowed with an action of the torus.

We will use an alternative description of the double loop groupoid. Its elements will be smooth maps

$$\phi\colon[0,1]^2\to M$$

together with commuting elements $g,h\in G$ such that $\phi(1,0)=\phi(0,0)g$, $\phi(0,1)=\phi(0,0)h$. Call this set $\mathcal{L}^2_{\langle g,h\rangle}M$ and take

$$\mathcal{L}^2M:=\bigsqcup_{\{(g,h)\in G^2|gh=hg\}}\mathcal{L}^2_{\langle g,h\rangle}M.$$

The natural action of conjugation by elements in G gives us the description: $\mathsf{L}^2\mathsf{X}\cong[(\mathcal{L}^2M)/G]$.

We consider the functional of double loops

$$\mathcal{H}(\phi) := \int_{[0,1]^2} (||\frac{d\phi}{ds}||^2 + ||\frac{d\phi}{dt}||^2) ds\, dt;$$

we will apply stationary phase approximation à la Witten-Atiyah to the Feynman integral

$$\int_{\mathsf{L}^2\mathsf{X}} e^{-i\mathcal{H}(\phi)}\mathcal{D}\phi.$$

We need to find the equivariant normal bundle on $\mathsf{L}^2\mathsf{X}$ to the fixed points of the action of $\mathbb{R}^2$, namely the ghost double loops.

For commuting $g, h \in G$, take the part of the groupoid of ghost double loops parameterized by $M^{\langle g,h\rangle}$, the fixed point set of the group generated by g and h. Call $\iota : M^{\langle g,h\rangle} \hookrightarrow M$ the inclusion, and suppose the the orbifold X is a complex orbifold (the pullback bundle ι^*TM can be locally simultaneously diagonalized with respect to the actions of g and h). Then one can write the total Chern class of ι^*TM as $\prod_j(1+x_j)$ such that the line bundle x_j comes provided with the action of the group $\langle g, h\rangle$ parameterized by the irreducible representation λ_j.

We are using the following fact about equivariant complex K-theory. If a group Γ acts trivially on a space Y, then

$$K^*_\Gamma(Y) \cong K^*(Y) \otimes R(\Gamma),$$

that is, the equivariant K-theory of Y is isomorphic to the ordinary K-theory of Y tensored with the representation ring of Γ. Then the equivariant Chern character associated to the $\langle g, h\rangle$ equivariant line bundle x_j is $ch_{\langle g,h\rangle}(x_j) = e^{x_j} \otimes \chi_{\lambda_j}$, where e^{x_j} is the Chern character of the line bundle and χ_{λ_j} is the character of the λ_i representation. As we have simultaneously diagonalized the actions of g and h, the character of an irreducible representation is determined by a root of unity associated to each g and h. So let $\sigma_j : \langle g, h\rangle \to [0, 1)$ be such that $\chi_{\lambda_j}(g) = e^{2\pi i\sigma_j(g)}$; then one can consider $(1 + x_j + 2\pi i\sigma_j)$ as the Chern class of the equivariant bundle x_j.

The equivariant Euler class of the normal bundle of the embedding of ghost double loops

$$M^{\langle g,h\rangle} \to \mathcal{L}^2_{\langle g,h\rangle}M$$

is then

$$\left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} \frac{1}{x_j} \right\} \left\{ \prod_j \prod_{(k,l)\in\mathbb{Z}^2} (x_j + l\hat{p} + k\hat{q} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}) \right\},$$

where $\hat{p}$ and $\hat{q}$ are formal variables that keep track of the fractional periods of each of the circles of the torus.

Applying the fixed point formula (3.2.1) of Ando-Morava [**3**], one obtains

$$p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1) = p^{M^{\langle g,h\rangle}} \left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j \right\} \left\{ \prod_j \prod_{(k,l)\in\mathbb{Z}^2} \frac{1}{x_j + l\hat{p} + k\hat{q} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}} \right\}.$$

Rearranging the expression in the second parenthesis by factoring $k\hat{q}$ and keeping the l fixed, the second parenthesis becomes:

$$\prod_{l\in\mathbb{Z}} \left(\prod_{k>0} \frac{1}{k^2\hat{q}^2} \right) \left((x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q}) \prod_{k>0} \left(1 - \frac{(x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q})^2}{k^2\hat{q}^2} \right) \right)^{-1}.$$

Renormalization (see [**5, 3**]) gives

$$\prod_{k>0} \frac{1}{k^2\hat{q}^2} = \frac{\hat{q}}{2\pi},$$

$$\prod_{k>0} \left(1 - \frac{(x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q})^2}{k^2\hat{q}^2}\right)^{-1} = \frac{\hat{q}}{2\pi} \frac{\frac{\pi}{\hat{q}}}{\sin\left(\frac{\pi}{\hat{q}}(x_j + l\hat{p} + \sigma_j(g)\hat{p} + \sigma_j(h)\hat{q})\right)}.$$

Replacing the variable $\hat{q}$ by its holonomy $2\pi i$, our push forward $p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1)$ becomes

$$p^{M^{\langle g,h\rangle}} \left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j \right\} \left\{ \prod_j \prod_{l\in\mathbb{Z}} \frac{\frac{1}{2}}{\sinh\frac{1}{2}(x_j + l\hat{p} + \sigma_j(g)\hat{p} + 2\pi i\sigma_j(h))} \right\};$$

pairing the hyperbolic sines of l and $-l$ one gets that

$$2\sinh\frac{(x_j + l\hat{p} + \sigma_j(g)\hat{p} + 2\pi i\sigma_j(h))}{2} 2\sinh\frac{(x_j - l\hat{p} + \sigma_j(g)\hat{p} + 2\pi i\sigma_j(h))}{2} =$$

$$\frac{1 - e^{-x_j - \sigma_j(g)\hat{p} - 2\pi i\sigma_j(h) - l\hat{p}}}{e^{-\frac{1}{2}(x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))} e^{-\frac{l}{2}\hat{p}}} \frac{e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}} - 1}{e^{\frac{1}{2}(x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h))} e^{-\frac{l}{2}\hat{p}}}.$$

As a result,

$$p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1) = p^{M^{\langle g,h\rangle}} \left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j \right\} \times$$

$$\left\{ \prod_j \frac{\frac{1}{2}}{\sinh\frac{1}{2}(x_j + \sigma_j(g)\hat{p} + 2\pi i\sigma_j(h))} \prod_{l>0} \frac{-e^{-\hat{p}l}}{(1 - e^{-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h)-l\hat{p}})(1 - e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}})} \right\}$$

$$= p^{M^{\langle g,h\rangle}} \left\{ \prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j \right\} (-e^{\hat{p}})^{\frac{1}{12}} \times$$

$$\left\{ \prod_{l>0,j} \frac{e^{\frac{1}{2}(-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h))}}{(1 - e^{-x_j-\sigma_j(g)\hat{p}-2\pi i\sigma_j(h)-(l-1)\hat{p}})(1 - e^{x_j+\sigma_j(g)\hat{p}+2\pi i\sigma_j(h)-l\hat{p}})} \right\}.$$

Making the change of variables $p = e^{-\hat{p}}$, assuming that the first Chern class of M satisfies $c_1(M) = 0$, i.e. $\prod_j e^{x_j} = 1$, and integrating over $M^{\langle g,h\rangle}$, we have that

$$p^{\mathcal{L}^2_{\langle g,h\rangle}M}(1) = \frac{p^{\left(-\frac{dim(M)}{12}+i\pi+\frac{\mathrm{age}(g)}{2}\right)} e^{-\pi i \mathrm{age}(h)} \left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j\right\}}{\prod_{l>0,j}(1 - p^{l-1+\sigma_j(g)}e^{-x_j-2\pi i\sigma_j(h)})(1 - p^{l-\sigma_j(g)}e^{x_j+2\pi i\sigma_j(h)})}[M^{\langle g,h\rangle}].$$

Adding all the fixed point data and averaging, one gets the orbifold elliptic genus:

$$Ell_{orb}([M/G]) =$$

$$\frac{1}{|G|} \sum_{gh=hg} \frac{p^{\left(-\frac{dim(M)}{12}+i\pi+\frac{\mathrm{age}(g)}{2}\right)} e^{-\pi i \mathrm{age}(h)} \left\{\prod_{\{j|\sigma_j(g)=\sigma_j(h)=0\}} x_j\right\}}{\prod_{l>0,j}(1 - p^{l-1+\sigma_j(g)}e^{-x_j-2\pi i\sigma_j(h)})(1 - p^{l-\sigma_j(g)}e^{x_j+2\pi i\sigma_j(h)})}[M^{\langle g,h\rangle}].$$

This coincides with the constant term in the y-expansion of the formula of Borisov-Libgober [**10, 16, 18**] except for a renormalization factor. One could use

a device like that of Hirzebruch [**21**] to recover the full formula. In any case the localization principle holds in this case.

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